

p -adic L -functions on Hida Families

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1 Overview

A major theme in the theory of p -adic deformations of automorphic forms is how p -adic L -functions over eigenvarieties relate to the geometry of these eigenvarieties. In this article we prove results in this vein for the ordinary part of the eigencurve (i.e. Hida families). We address how Taylor expansions of one variable p -adic L -functions varying over families can detect "bad" geometric phenomena: crossing components of a certain intersection multiplicity and ramification over the weight space. Our methods involve proving a converse to a result of Vatsal relating congruences between eigenforms to their algebraic special L -values and then p -adically interpolating congruences using formal models.

1.1 Congruences between Cusp Forms and Special Values

The connection between algebraic parts of special values and congruences between eigenforms was observed by Mazur. The underlying principle is that congruent forms should have congruent special values. General results were proven by Vatsal ([23]) in a number of different situations in this direction with applications towards nonvanishing theorems in mind. In particular, let $N > 3$ and $k > 1$. Let $p > 3$ be a prime and let K be an extension of \mathbb{Q}_p containing the Fourier coefficients of all normalized eigenforms for the congruence subgroup $\Gamma_1(N)$ and weight k . Let $\mathbb{T}_{N,k}$ be the Hecke algebra of $S_k(N, \mathcal{O}_K)$. A maximal ideal \mathfrak{m} of $\mathbb{T}_{N,k}$ corresponds to a residual Galois representation $\bar{\rho}$. We will make the assumption that

$$H_1(\mathbb{H}/\Gamma_1(N), \mathcal{L}_n(\mathcal{O}_K))_{\mathfrak{m}}^{\pm} \cong \mathbb{T}_{N,k,\mathfrak{m}}$$

as $\mathbb{T}_{N,k}$ -modules, where $\mathcal{L}_n(\mathcal{O}_K)$ is the local system associated to $L_n(\mathcal{O}_K)$. This isomorphism is unique up to an element in \mathcal{O}_K^* and a choice of isomorphism corresponds to choosing periods. If f and g are two eigenforms with residual representation $\bar{\rho}$, then we have two \mathcal{O}_K -algebra homomorphisms, δ_f and δ_g , from $\mathbb{T}_{N,k,\mathfrak{m}}$ to \mathcal{O}_K . Any congruence satisfied between f and g is necessarily satisfied between δ_f and δ_g . Evaluating δ_* on the appropriate cycle (maybe extending scalars to include the necessary roots of unity) yields special L -values. These special values must satisfy any congruence between f and g .

In this article we prove that the converse is true. We show that if periods Ω_f and Ω_g can be chosen so that the algebraic special values of both eigenforms are congruent mod p^r , then we have $f \cong g \pmod{p^r}$. To prove this result, we use the theory of modular symbols introduced by Manin [17] and generalized further by Ash and Stevens [2]. We show that a modular symbol is completely determined by its special values (in fact, finitely many special values are needed) and then use a standard congruence module argument (see [12] or [20]) with a Hecke module made up of modular

symbols. This result can be reinterpreted using the p -adic L -function constructed in [18]. By the uniqueness of the p -adic L -function, we conclude that we only need to consider the special values $L^{\text{alg}}(f, \chi, 1)$ as opposed to all critical values between 1 and k . Combining the results from the first half of the article with Vatsal's result gives:

Theorem 1.1. *Let f and g be eigenforms as above and assume that*

$$H_1(\mathbb{H}/\Gamma_1(N), \mathcal{L}_n(\mathcal{O}_K))_{\mathfrak{m}}^{\pm} \cong \mathbb{T}_{N,k,\mathfrak{m}}.$$

Then the following are equivalent:

- *The forms f and g are congruent modulo p^r*
- *There exists periods Ω_f^{\pm} and Ω_g^{\pm} such that for all Dirichlet characters χ we have*

$$\tau(\chi) \frac{L(f, \chi, 1)}{2\pi i \Omega_f^{\pm}} \cong \tau(\chi) \frac{L(g, \chi, 1)}{2\pi i \Omega_g^{\pm}} \pmod{p^r},$$

where $\tau(\chi)$ denotes the Gauss sum.

- *There exists $N > 0$ and periods Ω_f^{\pm} and Ω_g^{\pm} such that for all Dirichlet characters χ of character less than N we have*

$$\tau(\chi) \frac{L(f, \chi, 1)}{2\pi i \Omega_f^{\pm}} \cong \tau(\chi) \frac{L(g, \chi, 1)}{2\pi i \Omega_g^{\pm}} \pmod{p^r},$$

where $\tau(\chi)$ denotes the Gauss sum.

- *There exists p -adic L -functions defined using the same periods such that $L_p(f, \chi, s) - L_p(g, \chi, s)$ is divisible by p^r (here $L_p(f, \chi, s)$ denotes the cyclotomic p -adic L -function twisted by the Dirichlet character χ) for all χ .*

1.2 Crossing components in Hida families

In the second part of this article we prove a geometric analogue to the results of the first part. In the first part we are concerned with congruences between cusp forms of powers of p of level N . This corresponds to a geometric picture in $X := \text{Spec } \mathbb{T} \otimes \mathcal{O}_K$. The points of co-dimension zero in X correspond to cuspidal eigenforms of level N . The points of co-dimension one correspond to residual representation. Let x_f and x_g be co-dimension zero points of X corresponding to eigenforms f and g . Then x_f and x_g specialize to the same co-dimension one point x_ρ if and only if $f \cong g \pmod{\pi_K}$. We define the "intersection multiplicity" of the components $\overline{x_f}$ and $\overline{x_g}$ at x_ρ to be

$$\dim_{\mathcal{O}_K/\phi_K} \mathbb{T}_{x_\rho} / (\mathfrak{p}_f + \mathfrak{p}_g)$$

where \mathfrak{p}_* is the prime corresponding to x_* . This definition agrees with the algebraic definition provided in [8]. The largest power of π_K for which f and g are congruent is equal to this intersection multiplicity. The results from the first part may be reformulated to relate congruences between special L -values and the intersection multiplicity of $\overline{x_f}$ and $\overline{x_g}$.

This geometric interpretation of congruences between eigenforms suggests analogues for Hida families. Congruences between connected components of Hida families corresponds to crossing components in the generic fiber. Instead of looking at congruences for different powers of π_K we will be

interested in the intersection multiplicity of two crossing components. Any two such components correspond to minimal primes of $\mathbb{T}_{\mathfrak{m}}$ where \mathfrak{m} is a maximal prime of the big Hecke-Hida algebra.

The p -adic L -functions we will be interested can be described as follows. For a fixed Dirichlet character χ we will describe $L_p(\chi, s) \in \mathbb{T}_{\mathfrak{m}}$ that interpolates $L(f, \chi, 1)^{alg}$ as f varies over the hida family. This L -function is obtained by fixing the "twisted" cyclotomic variable in a slightly modified version of the p -adic L -function described in [7]. For a connected component C of $\text{Spec}(\mathbb{T}_{\mathfrak{m}})$ we may restrict $L_p(\chi, s)$ to C , which we call $L_p(C, \chi, s)$. If the natural projection onto the weight space $\pi : C \rightarrow \text{Spec} \mathcal{O}_K[[T]]$ is etale at a point x , then we may find a small enough affinoid neighborhood U around x so that $L_p(C, \chi, s)|_U$ may be written as a power series $f_{x, \chi, C}(T)$ in a canonical way. We may now state our main result:

Theorem 1.2. *Let C_1 and C_2 be two components of $\text{Spec}(\mathbb{T}_{\mathfrak{m}})^{rig}$ crossing at x . Assume that π is etale at x and that the weight of x is a limit of classical weights (e.g. any \mathcal{O}_K -valued weight.) Let I be the intersection multiplicity of C_1 and C_2 at x . Then $f_{x, \chi, C_1}(T)$ and $f_{x, \chi, C_2}(T)$ are congruent modulo $(T - \pi(x))^I$. In other words, the Taylor expansions of f_{x, χ, C_1} and f_{x, χ, C_2} agree for the first I terms. For some character χ' , the two L -functions differ at the $I + 1$ -th term.*

To prove this theorem, we reduce the problem to the situation where both components look *almost* like $\text{Spec}(\mathcal{O}_K[[T]])$. This involves choosing small affinoid neighborhoods of x in the rigid fiber and choosing an appropriate formal model in the sense of Raynaud [19]. This formal model is chosen in a way that allows us to remember congruences. When both components look like $\text{Spec}(\mathbb{Z}_p[[T]])$, we repeatedly apply the p -adic Weierstrass preparation theorem to further simplify the situation. Finally, we will apply Theorem 3.5 to a limit of classical weights approaching $\pi(x)$.

It would be interesting to extend these results to the positive slope part of the eigencurve. There is one technical difficulties that immediately come to mind. The construction of Coleman and Mazur [4] does not come with a formal integral model. The large Hecke-Hida algebra over the integers of a local field allows us to see congruences. Without an integral model that captures all congruences, our methods fail.

1.3 Ramification over the weight space

In the final section we describe the behavior of the p -adic L -functions at points on Hida families that are ramified above the weight space. Informally our result says that a component is etale over the weight space if and only if no poles are introduced when we differentiate each L -function along the weight space. More precisely, let C be a regular connected component of a Hida family and let T' be any parameter for our weight space $\text{Spec}(\Lambda)$. Our parameter defines a derivation on the function field of $\text{Spec}(\Lambda)$ denoted $\frac{d}{dT'}$. This derivation extends to the function field K of C . If C is etale over $\text{Spec}(\Lambda)$ then $\frac{d}{dT'}$ will give a derivation on the global functions A of C . If for some $x \in C$ there exists $f \in A$ such that $\frac{df}{dT'}$ has a pole at x , then x must be ramified over the weight space. The largest pole occurring will be one less than the ramification index. Our main result is that it is enough to check if there exists a Dirichlet character such that $\frac{dL_p(C, \chi)}{dT'}$ has poles.

Theorem 1.3. *A regular \mathcal{O}_K -point $x \in C^{rig}$ is ramified over $\pi(x) \in \text{Spec}(\Lambda_K)$ if and only if there exists a Dirichlet twist such that $\frac{d}{dT}L(C, \chi)$ has a pole at x , where T is a parameter of the*

weight space. The ramification index of x over $\pi(x)$ is equal to one more than the largest order pole occurring.

The proof of this theorem is similar to the proof of Theorem 1.2. We first take a small affinoid neighborhood around x which comes naturally equip with a formal model that is isomorphic to $\text{Spec}(\mathcal{O}_K\langle Y \rangle)$. This allows us to apply the p -adic Weierstrauss preparation theorem and Theorem 3.5.

1.4 Further Remarks

The results in this article should have generalizations to ordinary families of automorphic forms for larger algebraic groups. A several variable p -adic L -function was constructed by Dimitrov in [5] that varies over ordinary families of Hilbert modular forms. It seems likely that our geometric methods could be adapted to this context. Even more generally, it seems plausible that one could construct measures using compatible families of automorphic cycles living in Emerton's completed cohomology (see [6]) that detect ramification over the weight space and crossing.

The author is currently investigating the extension of the results in this paper to points of characteristic p . Following the philosophy of [1] we may view these points as the boundary of our Hida families. These points can be regarded as the ordinary part of the spectral halo conjectured by Coleman. In [1] a formal model is constructed for the part of the eigencurve living over the outer Halo of the weight space. It is plausible that p -adic L -functions can be constructed on the spectral halo and that this formal model could be used to imitate the techniques used in this paper.

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2 Modular Symbols and the Eichler-Shimura Isomorphism

2.1 Modular Symbols and Various Cohomology Theories

Throughout this article we will let $N > 3$ and $\Gamma = \Gamma_1(N)$. In particular, Γ is free of torsion. Let D_0 be the divisors of $\mathbb{P}(\mathbb{Q})$ of degree 0 with a natural left action of Γ . For any $Z[\Gamma]$ -module E , we let $\Phi(E) = \text{Hom}_\Gamma(D_0, E)$. These are modular symbols with values in E . If E also admits a right action of $GL_2(\mathbb{Q})$ (resp. $GL_2(\mathbb{Z})$) we may define a right action on $\Phi(E)$ of $GL_2(\mathbb{Q})$ (resp. $GL_2(\mathbb{Z})$). If $\alpha \in \Phi(E)$ and $g \in GL_2(\mathbb{Q})$ then $\alpha|_g$ sends $(r_1 - r_2)$ to $\alpha(g(r_1) - g(r_2))|g$.

It is known that $\Phi(E) \cong H_c^1(\mathbb{H}/\Gamma, \tilde{E})$ (see [2, Proposition 4.1],) where \tilde{E} is the local system associated to E . The group $H_!^1(\mathbb{H}/\Gamma, \tilde{E})$ is defined to be the image of $H_c^1(\mathbb{H}/\Gamma, \tilde{E})$ in $H^1(\mathbb{H}/\Gamma, \tilde{E})$. We may compare topological cohomology, which we may interpret as singular or deRham using deRham's theorem, and group cohomology to get the following commutative diagram whose vertical arrows are isomorphisms.

$$\begin{array}{ccccc}
\Phi(E) \cong H_c^1(\mathbb{H}/\Gamma, \widetilde{E}) & \longrightarrow & H_{\dagger}^1(\mathbb{H}/\Gamma, \widetilde{E}) & \longrightarrow & H^1(\mathbb{H}/\Gamma, \widetilde{E}) \\
\downarrow & & \downarrow & & \downarrow \\
& \longrightarrow & H_{\dagger}^1(\Gamma, E) & \longrightarrow & H^1(\Gamma, E).
\end{array}$$

We call ϕ the map that takes a modular symbol to a 1-cocycle. Explicitly this map takes the modular symbol α to the 1-cocycle that sends g to $\alpha(g(x)) - \alpha(x)$ for some $x \in \mathbb{P}(\mathbb{Q})$. The right two vertical maps send a 1-form to ω to the 1-cocycle

$$g \rightarrow \int_{z_0}^{g(z_0)} \omega,$$

where z_0 can be any number in \mathbb{H} . If ω is compact we may even allow z_0 to be in $x \in \mathbb{P}(\mathbb{Q})$. When E is not a \mathbb{R} -vector space we must use singular cohomology, but the idea is essentially the same. A more thorough explanation can be found in the appendix of [11].

2.2 The Complex Conjugation Involution

Each space in the above diagram has an action induced by the involution σ of \mathbb{H} given by $z \rightarrow -\bar{z}$. Consider the 1-cocycle β defined by a 1-form ω_β . Then we get a new 1-cocycle that sends g to

$$\int_i^{-\overline{g(i)}} \omega_\beta = \int_i^{g(i)} \sigma^*(\omega_\beta).$$

Thus β is sent to the 1-cocycle $g \rightarrow \beta(\xi g \xi^{-1})$, where $\xi = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. On deRham cohomology the form ω is sent to its pullback $\sigma^*(\omega)$ under σ . In particular, holomorphic forms are sent to anti-holomorphic forms and vice versa. The action on modular symbols sends $\alpha \in \Phi(L_n(\mathbb{C}))$ to $\alpha|_\xi$. It is readily checked that all of these actions are compatible with the maps in the above diagram. It is also clear that we have an action on cohomology (resp. modular symbols) with coefficients in $L_n(A)$ where A is any ring in \mathbb{C} .

If V is any one of the spaces in the diagram above, we have a decomposition $V = V^+ \oplus V^-$. Here V^+ are the elements of V fixed by the action of complex conjugation and V^- are the elements negated by this action. In fact if 2 is invertable in a subring A of \mathbb{C} then we get a diagram whose vertical arrows are isomorphisms.

$$\begin{array}{ccccc}
\Phi(L_n(A))^\pm \cong H_c^1(\mathbb{H}/\Gamma, \widetilde{L_n(A)})^\pm & \longrightarrow & H_{\dagger}^1(\mathbb{H}/\Gamma, \widetilde{L_n(A)})^\pm & \longrightarrow & H^1(\mathbb{H}/\Gamma, \widetilde{L_n(A)})^\pm \\
\downarrow & & \downarrow & & \downarrow \\
& \longrightarrow & H_{\dagger}^1(\Gamma, L_n(A))^\pm & \longrightarrow & H^1(\Gamma, L_n(A))^\pm.
\end{array}$$

2.3 The Eichler Shimura Isomorphism

Let $f \in S_{n+2}(\Gamma)$ and let L_n be the space of homogeneous polynomials in x and y of degree n . The 1-form $f(z)(x - zy)^n dz$, which takes values in L_n is invariant under Γ and therefore gives us an element ω_f of $H^1(\mathbb{H}/\Gamma, \widetilde{L_n}(\mathbb{C}))$. The form ω_f does not have compact support, but it turns out that $\omega_f \in H^1_!(\mathbb{H}/\Gamma, \widetilde{L_n}(\mathbb{C}))$. It turns out that if we take the real part of ω_f we get an isomorphism

$$S_{n+2}(\Gamma) \cong H^1_!(\mathbb{H}/\Gamma, \widetilde{L_n}(\mathbb{R})) \cong H^1_p.$$

We easily check that the action of complex conjugation sends $f(z)(x - zy)^n dz$ to $-f(-\bar{z})(x + \bar{z}y)^n d\bar{z}$. In particular we see that the projection of ω_f onto $H^1(\mathbb{H}/\Gamma, \widetilde{L_n}(\mathbb{C}))^\pm$ is

$$\frac{f(z)(x - zy)^n dz \pm -f(-\bar{z})(x + \bar{z}y)^n d\bar{z}}{2}.$$

2.4 Integral Cohomology

Let f be an eigenform for Γ and let ω_f^\pm be the corresponding 1-form in $H^1(\mathbb{H}/\Gamma, \widetilde{L_n}(\mathbb{C}))^\pm$. We define a modular symbol α_f^\pm by

$$\alpha_f^\pm(\{r_1\} - \{r_2\}) = \int_{r_1}^{r_2} \omega_f^\pm.$$

This gives a Hecke equivariant map $s : S_k(\Gamma) \rightarrow \Phi(L_n(\mathbb{C}))^\pm$. By a theorem of Shimura (see [9, Theorem 4.8]) the subspace of $\Phi(L_n(\mathbb{C}))^\pm$ that has the same Hecke eigenvalues as f is one dimensional. If A is a subring of \mathbb{C} containing the Hecke eigenvalues of f , there exists periods Ω_f^\pm such that

$$\frac{\alpha_f^\pm}{\Omega_f^\pm} \in \Phi(L_n(A))^\pm.$$

The map s is a section of the map $\phi : \Phi(L_n(\mathbb{C}))^\pm \rightarrow H^1(\Gamma, \widetilde{L_n}(\mathbb{C}))^\pm$. We know that $\phi(L_n(A))^\pm \subset H^1(\Gamma, \widetilde{L_n}(A))^\pm$ by our explicit description of ϕ .

3 Congruences Between Cusp Forms and L-functions

In this section we prove that two cusp forms are congruent if and only if the "algebraic" special values of their L -functions admit congruences for all twists. It turns out that we only need to consider finitely many twisted L -functions to determine if there are congruences.

3.1 Special Values of Modular Symbols

Let $[\frac{p}{q}]$ denote the degree zero divisor $\{\frac{p}{q}\} - \{\infty\}$. For a primitive Dirichlet character χ of conductor D we define

$$\Lambda(\chi) = \sum_{m=0}^{D-1} \overline{\chi(m)} \left[\frac{p}{q}\right] \in D_0 \otimes \mathbb{Z}[\chi].$$

Let E is a $\mathbb{Z}_p[\Gamma]$ -module and let $\alpha \in E$. Define the special value $L(\alpha, \chi)$ of α to be $\alpha(\Lambda(\chi))$. If α is the modular symbol associated to a cusp form then the special values of α are related to the special values of the forms L -functions. The next proposition says that a modular symbol is completely determined by special values and that we have some control over which special values we need to look at. More specifically, let $\epsilon > 0$ and let $r \in \mathbb{Z}$ be prime to p . Take X to be the set of primes q larger than ϵ that satisfy the congruences

$$\begin{aligned} q &\equiv r \pmod{p} \\ q &\equiv 1 \pmod{N^2}. \end{aligned}$$

We will prove that α is determined by its special values for Dirichlet characters with conductor in X . This type of result was first observed by Stevens (Lemma 2.1 in [22]) for weight 2 forms.

Lemma 3.1. *Let $\frac{c}{d}$ be a reduced fraction whose denominator is $1 \pmod{N}$. There exists $\gamma \in \Gamma$ such that the denominators of $\gamma(\frac{c}{d})$ and $\gamma(0)$ are in X .*

Proof. First assume that $p \nmid c$. If this is not the case, replace $\frac{c}{d}$ with

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \frac{c}{d} = \frac{c+d}{d},$$

whose denominator is prime to p . A similar manipulation will guarantee c is coprime to N as well. Let l_1 be a prime number that is congruent to 1 modulo N^2 and congruent to r modulo p . We may take l_1 large enough to be contained in X and so that $l_1 \nmid c$. Next take z to be a prime satisfying

$$\begin{aligned} z &\equiv 1 \pmod{N^2}, \\ z &\equiv dl_1 \pmod{Nc}, \text{ and} \\ z &\equiv r \pmod{p}. \end{aligned}$$

Then z can be written as $yNc + dl_1$. We set $l_2 = Ny$. There is a matrix $\gamma = \begin{pmatrix} t_1 & t_2 \\ l_1 & l_2 \end{pmatrix}$ in Γ and we may assume that t_1 is divisible by z . We see that γ satisfies the desired properties. \square

Lemma 3.2. *Let α be a module symbol with values in E and assume that α doesn't map to zero in $H^1(\Gamma, E)$. Let r be prime to p with $r \not\equiv 1 \pmod{p}$ and let $\epsilon > 0$. Then either $L(\alpha, \chi_{triv}) \neq 0$ or there exists a primitive Dirichlet character χ whose conductor is in X such that $L(\alpha, \chi) \neq 0$.*

Proof. First, extend any modular symbol in $\Phi(E)$ to an element of $\text{Hom}_\Gamma(D_0 \otimes_{\mathbb{Z}} \overline{\mathbb{Z}_p}, E \otimes_{\mathcal{O}} \overline{\mathbb{Z}_p})$. For any subset S of \mathbb{Q} , we will define

$$\begin{aligned} A(S) &:= \bigoplus_{x \in S} \overline{\mathbb{Z}_p}[x], \\ A'(S) &:= \text{Hom}(A(S), E \otimes_{\mathcal{O}} \overline{\mathbb{Z}_p}), \\ \rho_S &:= \text{Hom}_\Gamma(D_0 \otimes_{\mathbb{Z}} \overline{\mathbb{Z}_p}, E \otimes_{\mathcal{O}} \overline{\mathbb{Z}_p}) \rightarrow A'(S), \end{aligned}$$

where ρ_S is the natural map. Note that the map $\rho_{\mathbb{Q}}$ is an injection and that if $S \subset T \subset \mathbb{Q}$ then ρ_S factors through ρ_T . Let S_X be the set of rational numbers whose denominator is in X . Let

$S'_X = S_X \cap (0, 1]$. We will be interested in ρ_{S_X} and $\rho_{S'_X}$.

Let α be a modular symbol with $L(\alpha, \chi) = 0$ for all χ whose conductor is in X . For $m \in X$, we let S_m be the rational numbers $\frac{1}{m}, \dots, m-1m$. The $\overline{\mathbb{Z}_p}$ -span of $\Lambda(\chi)$ for all χ of conductor m is a submodule M of $A(S_m)$. Then M consists of all elements $\sum a_i \frac{i}{m}$ where $\sum a_i = 0$. That is to say, M is the kernel of the augmentation map on $A(S_m)$. Consider $[\frac{N}{m}] \in A(S_m)$. We have

$$\begin{aligned} \alpha([\frac{N}{m}]) &= \alpha([\frac{N}{m}]) - \alpha([0]) \\ &= \alpha(\{\frac{N}{m}\} - \{0\}), \end{aligned}$$

where $\alpha([0]) = 0$ because $L(\alpha, \chi_{triv}) = 0$. Let $\gamma_0 = \begin{pmatrix} 1 & 0 \\ -Nk & 1 \end{pmatrix}$, where $m = N^2k + 1$. Then

$$\begin{aligned} \gamma_0(\{\frac{N}{m}\} - \{0\}) &= \{-N\} - \{0\} \\ &= [-N] - [0]. \end{aligned}$$

There is a parabolic element $\gamma_1 \in \Gamma$ such that $\gamma_1([N]) = [0]$, so $\alpha([-N]) = 0$. It follows that $\alpha(\gamma_0(\{\frac{N}{m}\} - \{0\})) = 0$ and thus $\alpha(\{\frac{N}{m}\} - \{0\}) = \alpha([\frac{N}{m}]) = 0$. Since the span of M and $[\frac{N}{m}]$ is all of $A(S_m)$ we see that α is zero on all of $A(S_m)$. In other words $\rho_{S_m} = 0$.

Let $x \in S_X$. There is a parabolic element $\gamma' \in \Gamma$ such that $\gamma'(x) \in S'_X$. Applying γ'^{-1} to the equation $\alpha(\gamma'([x])) = 0$ shows that $\alpha([x]) = 0$. It follows that $\rho_{S_X}(\alpha) = 0$. Now we consider the divisor $\{0\} - \{\frac{c}{d}\}$, where $\frac{c}{d} = \gamma(0)$ for some $\gamma \in \Gamma$. The denominator c is $1 \pmod{N}$, so by 3.1 there exists $\gamma_0 \in \Gamma$ such that $\gamma_0(0)$ and $\gamma_0(\frac{c}{d})$ are in S_X . Then we have

$$\begin{aligned} \alpha(\{\gamma_0(0)\} - \{\gamma_0(\frac{c}{d})\}) &= \alpha([\gamma_0(0)] - [\gamma_0(\frac{c}{d})]) \\ &= 0, \end{aligned}$$

since α is in the kernel of ρ_{S_X} . This leads to a contradiction, since the cohomology class of α in $H^1(\Gamma, E)$ is represented by the 1-cocycle that sends γ to $\alpha(\{0\} - \{\gamma(0)\})$. □

Corollary 3.3. *Let \mathcal{O} be as in the lemma. Let α_1 and α_2 be modular symbols that takes values in $L_n(\mathcal{O})$ and let I be an ideal of \mathcal{O} . The following are equivalent.*

- For r, ϵ and X be as in the lemma

$$L(\alpha_1, \chi) \equiv L(\alpha_2, \chi) \pmod{I}$$

for all χ with conductor in X .

- $\alpha_1 \equiv \alpha_2 \pmod{IL_n(\mathcal{O})}$.

Proof. By (1.16) of [12] we know that $\Phi(L_n(\mathcal{O})) \otimes \mathcal{O}/I \cong \Phi(L_n(\mathcal{O}/I))$. Consider the image β of the modular symbol $\alpha_1 - \alpha_2$ in $\Phi(L_n(\mathcal{O}/I))$. If $L(\alpha_1, \chi) \equiv L(\alpha_2, \chi) \pmod{I}$ then $L(\beta, \chi) = 0$. Then from the lemma, we see the first condition implies the second. The other direction is immediate. □

3.2 Special Values of L-functions

Let $f(z) \in S_k(\Gamma)$ and write $f(z) = \sum a_n q^n$ where $q = e^{2\pi iz}$. Then $L(s, f)$ is defined to be $\sum a_n n^{-s}$. We have

$$\int_0^{i\infty} f(z) z^m dz = \frac{m! L(m+1, f)}{(-2\pi i)^{m+1}}.$$

More generally, if χ is a Dirichlet character of conductor D we define $L(s, f, \chi)$ as $\sum a_n \chi(n) n^{-s}$. This yields

$$\sum_{a=0}^{D-1} \overline{\chi(a)} \int_{\frac{a}{D}}^{i\infty} f(z) z^m dz = \tau(\overline{\chi}) \binom{k-2}{m} m! \frac{L(m+1, f, \chi)}{(-2\pi i)^{m+1}}.$$

These two integrals tell us the special values of the modular symbol $s(\omega_f)$. In particular

$$s(\omega_f)(\Lambda(\chi)) = (\dots, \tau(\overline{\chi}) \binom{k-2}{m} m! \frac{L(m+1, f, \chi)}{(-2\pi i)^{m+1}}, \dots).$$

3.3 Congruences Between Special Values

In this section we use Theorem 3.2 in conjunction with a standard congruence module argument (cf [20] and [12]). Let f and g be normalized eigenforms in $S_k(\Gamma)$ with $f(z) = \sum a_n q^n$ and $g(z) = \sum b_n q^n$. Let K be a finite extension of \mathbb{Q} containing the eigenvalues of both eigenforms. Let v be a prime of \mathcal{O}_K whose residue characteristic is p . Let $R = \mathfrak{O}_{K(v)}$ be the localization of \mathcal{O}_K at v and let π_v be a uniformizer of R . Since 2 is invertible in R we have

$$\Phi(L_n(R)) \cong \Phi(L_n(R))^+ \oplus \Phi(L_n(R))^-.$$

Let $M^+ = (\mathbb{C}\alpha_f^+ \oplus \mathbb{C}\alpha_g^+) \cap \Phi(L_n(R))^+$. We remark that M^+ is invariant under the Hecke operators and has dimension 2. Let $M_*^+ = M^+ \cap \mathbb{C}\omega_*^+$ and let M^{*+} be the projection of M^+ onto $\mathbb{C}\omega_*^+$ (here $*$ can be f or g). The spaces M_*^+ and M^{*+} are free R -modules of rank one and $M_*^+ \subset M^{*+}$. Picking a basis of M_*^+ is equivalent to choosing a period Ω_*^+ . The choice of such a period is canonical up to a v -adic unit as discussed in [23]. Let $\beta_*^+ = \frac{\alpha_*^+}{\Omega_*^+}$ be the normalized modular symbol. Let us assume that we were able to choose the periods so that

$$L(\beta_f^+, \chi) \cong L(\beta_g^+, \chi) \pmod{v^m}$$

for all χ with conductor in X . Then by the previous section we know that $\beta_f^+ - \beta_g^+ \in v^m \Phi(L_n(R))$. In particular, we may find $x \in \Phi(L_n(R))$ with $\pi_v^m x = \beta_f^+ - \beta_g^+$. As x is in the space spanned by f^+ and g^+ we see that M^+ contains x . Thus we have an injection

$$R/\pi_v^m R \rightarrow \frac{M^+}{M_f^+ \oplus M_g^+}, \text{ defined by}$$

$$1 \pmod{\pi_v^m R} \rightarrow x \pmod{M_f^+ \oplus M_g^+}.$$

We get the following equivalence of Hecke modules:

$$\frac{M^{f+}}{M_f^+} \cong \frac{M^{f+} \oplus M^{g+}}{M^+} \cong \frac{M^{g+}}{M_g^+}, \text{ and}$$

$$\frac{M^{f+} \oplus M^{g+}}{M^+} \cong \frac{M^+}{M_f^+ \oplus M_g^+}.$$

In particular we find that

$$\frac{M^{f+}}{M_f^+} \otimes R/\pi_v^m R \cong_{\mathbb{T}} \frac{M^{g+}}{M_g^+} \otimes R/\pi_v^m R,$$

and both of these Hecke modules are equal to $R/\pi_v^m R$ as an R -module. The Hecke operator T_n acts on $\frac{M^{f+}}{M_f^+} \otimes R/\pi_v^m R$ (resp $\frac{M^{g+}}{M_g^+} \otimes R/\pi_v^m R$) through scalar multiplication by $a_n \bmod \pi_v^m R$ (resp $b_n \bmod \pi_v^m R$). The isomorphism of Hecke modules then implies $a_b \cong b_n \bmod \pi_v^m R$.

Theorem 3.4. *Let f and g be normalized eigenforms in $S_k(\Gamma)$. If there exists periods Ω_f^{\pm} and Ω_g^{\pm} (these are defined canonically up to p -adic unit) that satisfy*

$$(\dots, \tau(\overline{\chi}) \binom{k-2}{m} m! \frac{L(m+1, f, \chi)}{(-2\pi i)^{m+1} \Omega_f^{\pm}}, \dots) \cong (\dots, \tau(\overline{\chi}) \binom{k-2}{m} m! \frac{L(m+1, g, \chi)}{(-2\pi i)^{m+1} \Omega_g^{\pm}}, \dots) \bmod \pi_v^m,$$

for all characters whose conductor is in X then $f \cong g \bmod \pi_v^m$.

3.4 The one variable cyclotomic p -adic L -function

We will now describe the construction of cyclotomic p -adic L -function as described in [7]. The only difference in what we describe is that we allow a fixed tame level of the Dirichlet twists that we interpolate, while [7] only covers the tame level 1 case. This construction is more or less equivalent to the function described in [18]. Let f be an p -ordinary eigenform of weight $k \geq 2$ and level Np^r . We define $\mathbb{T}_{N,r,k}$ to be the Hecke algebra over \mathcal{O}_K generated by T_l for $l \nmid Np$, U_l for $l \mid Np$, and the diamond operators $\langle a \rangle$ for $a \bmod N$. Let \mathfrak{m} be the maximal ideal of $\mathbb{T}_{N,r,k}$ corresponding to the residue of f modulo p . We will assume that that

$$H_1(\mathbb{H}/\Gamma, \widetilde{L_{k-2}}(\mathcal{O}_K))_{\mathfrak{m}}^{ord} \cong (\mathbb{T}_{N,r,k})_{\mathfrak{m}}.$$

This will be true if the residual representation is p -distinguished.

There is a natural map from $(\mathbb{T}_{N,r,k})_{\mathfrak{m}}$ to \mathbb{C} sending each Hecke operator to its eigenvalue on f . Call this map δ_f and note that the image of δ_f is in $\mathcal{O}_K[f]$. We also have a natural map from $H_1(\mathbb{H}/\Gamma, \widetilde{L_{k-2}}(\mathcal{O}_K))_{\mathfrak{m}}^{ord}$ to \mathbb{C} that is induced by integrating each cycle along the differential ω_f . These maps are the same up to a complex period. This is the same period we encountered in the previous section (up to a p -adic unit.) The choice of period will be determined by the choice of isomorphism from our homology group to our Hecke algebra.

Let $M > 1$ be prime to p . We will let $\Lambda = \mathbb{Z}_p[[T]]$ be the standard Iwasawa algebra. Define Λ_M as $\Lambda[\mathbb{Z}/pM\mathbb{Z}^{\times}]$. Then $\Lambda[\mathbb{Z}/pM\mathbb{Z}^{\times}] \cong \varprojlim \mathbb{Z}_p[\mathbb{Z}/Mp^s\mathbb{Z}^{\times}]$ where $1+T$ goes to the topological generator $1+p$ of $1+p\mathbb{Z}_p$. Recall that for any \mathbb{Z}_p module A we may think of elements of $\Lambda_M \otimes A$ as measures on

$\mathbb{Z}_p^\times \oplus \mathbb{Z}/M\mathbb{Z}^\times$ with values in A . We define an element $L_{M,m}$ of $\Lambda_M \otimes H_1(\mathbb{H}/\Gamma, \widetilde{L_{k-2}}(\mathcal{O}_K))$ by the measure sending the open set $(a+p^r\mathbb{Z}_p, a+M\mathbb{Z})$ in $\mathbb{Z}_p^\times \oplus \mathbb{Z}/M\mathbb{Z}$ to $U_p^{-r} \{ \frac{a}{p^r M}, \infty \} \in H_1(\mathbb{H}/\Gamma, \widetilde{L_{k-2}}(\mathcal{O}_K))$. By the definition of U_p we see that this definition is a well defined measure (in particular it is additive).

When we specialize $L_{M,m}$ by δ_f we obtain an element $L_{M,m,f}$ of $\Lambda_M \otimes \mathcal{O}_K[f]$. Specializing at certain \mathbb{C}_p points of $\Lambda_M \otimes \mathcal{O}_K[f]$ will then give us special values of $L(f, \chi, s)$, for a primitive Dirichlet character of tame level M . Giving a \mathbb{C}_p point of $\Lambda_M \otimes \mathcal{O}_K[f]$ is equivalent to giving an element of $\text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times \oplus \mathbb{Z}/M\mathbb{Z}^\times, \mathbb{C}_p^\times)$. We may interpret any primitive Dirichlet character χ of tame level M as an character in $\text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times \oplus \mathbb{Z}/M\mathbb{Z}^\times, \mathbb{C}_p^\times)$. Then evaluating $L_{M,m,f}$ at this character corresponding to χ gives $\tau(\chi) \frac{L(f, \chi, 1)}{\Omega_f}$. If we multiply χ by the character $(1+p) \rightarrow (1+p)^s$, then $L_{M,m,f}$ evaluates to $\tau(\chi) s! \frac{L(f, \chi \omega^s, s+1)}{(2\pi i)^{s+1} \Omega_f}$ for $0 \leq s < k-1$ where ω is the Tichmuller character. Note that $\text{Spec}(\Lambda_M)$ is equal to $\phi(pM)$ copies of the open unit p -adic ball. One copy for each character of $\mathbb{Z}/pM\mathbb{Z}$. For a character ψ of conductor pM we let $L_{\psi,*}$ denote the restriction of $L_{M,m,*}$ to the unit ball corresponding to ψ .

Theorem 3.5. *Let f and g be two p -ordinary eigenforms of weight $k \geq 2$ and level Np^r . Let \mathcal{O}_K be the ring of integers of an extension of \mathbb{Q}_p that contains the coefficients of f and g . Let π be a uniformizer of \mathcal{O}_K . There exists an M' depending on the weight and level such that the following are equivalent*

- The forms f and g are congruent modulo π^t .
- The p -adic L -functions $L_{M,m,f}$ and $L_{M,m,g}$ are congruent modulo π^t for all M .
- The p -adic L -functions $L_{\psi,f}$ and $L_{\psi,g}$ are congruent modulo π^t for all primitive Dirichlet characters ψ .

Proof. That congruent forms have congruent p -adic L -functions follows from the above discussion. It was originally proven by Vatsal in [23] using the p -adic Weierstrass preparation theorem. That congruent p -adic L -functions implies a congruence of forms is a restatement of the results from the previous section. \square

Corollary 3.6. *Using the notation of the Theorem 3.5 the forms f and g are congruent modulo π^t if and only if for every Dirichlet character χ we have*

$$\tau(\chi) \frac{L(f, \chi, 1)}{\Omega_f} \equiv \tau(\chi) \frac{L(g, \chi, 1)}{\Omega_g} \pmod{\pi^t}.$$

Remark. *Theorem 3.4 tells us that the critical values $1, \dots, k-1$ of the twisted L -functions can detect congruences. This corollary says that it is sufficient to look at the critical value at $s = 1$.*

Proof. The necessity is already established. Conversely, let us assume that the congruence between the critical values at $s = 1$ holds for all characters. By Theorem 3.5 is enough to show that $\pi^t | L_{\psi,f} - L_{\psi,g}$, where we view $L_{\psi,*}$ as an element of $\mathcal{O}_K[[T]]$. By the Weierstrass preparation theorem we may write

$$L_{\psi,f} - L_{\psi,g} = \pi^r u(T) P(T)$$

where $u(T)$ is a unit and $P(T)$ is distinguished. Now let χ be a primitive character of conductor p^s . The \mathbb{C}_p -point of $\text{Spec}(\mathcal{O}_K[[T]])$ corresponding to χ sends T to $1 - \zeta_{p^s}$ for some primitive p^s -th root of unity. The p -adic valuation of $1 - \zeta_{p^s}$ is $\frac{1}{\phi(p^s)}$. As $P(T)$ is distinguished we know that $v_p(p(1 - \zeta_{p^s})) = \deg(P(T))\frac{1}{\phi(p^s)}$ if s is sufficiently large. Putting this together with our hypothesis gives

$$tv_p(\pi) \leq v_p(\pi^r u(1 - \zeta_{p^s})P(1 - \zeta_{p^s})) \quad (1)$$

$$= rv_p(\pi) + \deg(P(T))\frac{1}{\phi(p^s)}, \quad (2)$$

for sufficiently large s . Letting s tend to ∞ shows that $t \leq r$. □

4 p -adic L -functions on Hida families

In this section we describe a p -adic L -function that varies analytically over Hida families whose residual representation are p -stabilized. We will also interpret the results of the previous section in terms of p -adic L -functions. Throughout the remainder of this article we set K to be a finite extension of \mathbb{Q}_p and \mathcal{O}_K to be the ring of integers in K with uniformizing element π_K . We let $\Lambda_K := \Lambda \otimes \mathcal{O}_K$ be the Iwasawa algebra with coefficients in \mathcal{O}_K .

4.1 Hida Families

In this section we give a summary of Hida theory. For an introduction to the theory with tame level 1 see [11]. For the general situation see [13]. For the most part we adopt the notation of [7]. Let $M > 0$ be relatively prime to p . For $k \geq 2$, let $S_k(Np^\infty, \mathcal{O}_K)$ be the union of all weight k cusp forms for $\Gamma_1(Np^r)$ that are defined over the \mathbb{Z}_p -algebra \mathcal{O}_K . There is a natural action of $\mathbb{Z}/p^r\mathbb{Z}^\times$ on $S_k(Np^r\mathcal{O}_K)$ given by the product of the Nebentypus action and the character $\gamma \rightarrow \gamma^k$. These actions are compatible with the inclusion of $S_k(Np^r, \mathcal{O}_K)$ in $S_k(Np^{r+s}, \mathcal{O}_K)$ for any $s > 0$ so that we have an action of \mathbb{Z}_p^\times on $S_k(Np^\infty, \mathcal{O}_K)$. In particular $S_k(Np^\infty, \mathcal{O}_K)$ is a $\mathcal{O}_K[[\mathbb{Z}_p^\times]]$ -module and a Λ_K -module.

For a prime \mathfrak{p} of Λ_K of height one we write $\mathcal{O}(\mathfrak{p}) := \Lambda_K/\mathfrak{p}$. We say that \mathfrak{p} is classical of weight k if the residue has characteristic 0 and if the composition $\kappa_{\mathfrak{p}} := 1 + p\mathbb{Z}_p\Lambda_K \rightarrow \mathcal{O}(\mathfrak{p})$ coincides with the character $\lambda \rightarrow \lambda^k$ on an open subgroup of $1 + p\mathbb{Z}_p$.

Let \mathbb{T}_N be the $\mathcal{O}_K[[\mathbb{Z}_p^\times]]$ -algebra generated by the Hecke operators and diamond operators acting on $S_k(Np^\infty, \mathcal{O}_K)^{\text{ord}}$. Then \mathbb{T}_N is a Λ_K -algebra. A height one prime ideal \mathfrak{p} of \mathbb{T}_N is said to be classical of weight k if it lies above a weight k prime of Λ . Just as before, we set $\mathcal{O}(\mathfrak{p}) := \mathbb{T}_N/\mathfrak{p}$ and define $\kappa_{\mathfrak{p}}$ to be the character from $1 + p\mathbb{Z}_p$ to $\mathcal{O}(\mathfrak{p})$. The fibers of the residues of weight k primes of Λ_K in \mathbb{T}_N recover the Hecke algebra acting on weight k cusp forms of tame level N . More specifically, Hida proved the following theorem in [13].

Theorem 4.1. *Using the above notation we have:*

- \mathbb{T}_N is free of finite rank over Λ_K .

- Let $\mathfrak{p} \in \text{Spec}(\Lambda_K)$ be a classical height one prime of weight k . Then $\mathbb{T}_N \otimes \mathcal{O}(\mathfrak{p})$ is equal to the full Hecke algebra acting on $S_k(Np^\infty, \mathcal{O}(\mathfrak{p}))^{ord}[\kappa_{\mathfrak{p}}]$.

The Hecke algebra \mathbb{T}_N is a semi-local ring. The different maximal ideals correspond to the different representations of $\text{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ into $GL_2(\overline{\mathbb{F}}_p)$ that arise as residues of representations associated to cusp forms of tame level N . The classical height one primes of \mathbb{T}_N are in one to one correspondence with the Galois conjugacy classes of eigenforms in $S_k(Np^\infty, \mathcal{O}_K)$. The minimal primes represent maximal families of eigenforms. In particular, Let \mathfrak{a} be a minimal prime of \mathbb{T}_N . Consider the formal power series

$$f(q) = \sum T_n q^n \text{ with } T(n) \in \mathbb{T}_N/\mathfrak{a}.$$

Then the image of $f(q)$ in $\mathbb{T}_N/\mathfrak{a} \otimes \mathcal{O}(\mathfrak{p})$, where \mathfrak{p} is a classical height one prime, will give the Fourier expansion of the modular form corresponding to \mathfrak{p} .

4.2 The two variable cyclotomic p -adic L -function

Let \mathfrak{m} be a maximal prime of \mathbb{T}_N . Then the classical height one primes of $\mathbb{T}_{N,\mathfrak{m}}$ (the localization of \mathbb{T}_N at the prime \mathfrak{m}) correspond to eigenforms with the same residual representation. Equivalently, the Fourier coefficients of all classical height one primes of $\mathbb{T}_{N,\mathfrak{m}}$ are equivalent in the appropriate residue field. From now on we will assume that this residual representation is p -distinguished and irreducible. This is a necessary condition to use the previous section.

Theorem 4.2. *Let \mathfrak{m} be a maximal prime of $\mathbb{T}_{N,r,k}$ whose residual representation is irreducible and p -distinguished. Then $H_1(\mathbb{H}/\Gamma, \widetilde{L_{k-2}}(\mathcal{O}_K))_{\mathfrak{m}}^{\pm ord} \cong (\mathbb{T}_{N,r,k})_{\mathfrak{m}}$ as $\mathbb{T}_{N,r,k}$ -modules.*

Proof. [7, Proposition 3.1.1] □

We define $H_1(Np^\infty, \mathcal{O}_K) := \varprojlim H_1(\mathbb{H}/\Gamma_1(Np^r), \mathcal{O}_K)^{ord}_{\mathfrak{m}}$. It is proven in [7] that $H_1(Np^\infty, \mathcal{O}_K)^{\pm ord}$ is a free rank 1 \mathbb{T}_N -module. We also have

Theorem 4.3. *Let $\mathfrak{p}_{N,r,k}$ be the product of all primes of weight k in \mathbb{T}_N . Then*

$$H_1(Np^\infty, \mathcal{O}_K) \otimes \mathbb{T}_N/\mathfrak{p}_{N,r,k} \cong H_1(\mathbb{H}/\Gamma_1(Np^r), \widetilde{L_{k-2}}(\mathcal{O}_K))_{\mathfrak{m}}^{ord}.$$

Proof. This is [14, Theorem 1.9] □

As in the previous section, we define a measure sending the open set $(a + p^r \mathbb{Z}_p, a + M\mathbb{Z})$ in $\mathbb{Z}_p^\times \oplus \mathbb{Z}/M\mathbb{Z}$ to $U_p^{-r} \{ \frac{a}{p^r M}, \infty \} \in H_1(Np^\infty, \mathcal{O}_K)$. This defines an element

$$L_p(\mathbb{N}, \mathfrak{m}, M) \in H_1(Np^\infty, \mathcal{O}_K) \otimes \Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times].$$

Fix an isomorphism between \mathbb{T}_N and $H_1(Np^\infty, \mathcal{O}_K)$. This gives

$$L_p(\mathbb{N}, \mathfrak{m}, M) \in \mathbb{T}_{\mathfrak{m},N} \otimes \Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times].$$

By specializing to a classical weight one prime of $\mathbb{T}_{\mathfrak{m},N}$ corresponding to an eigenform f we recover the p -adic L -function $L_{M,\mathfrak{m},f}$ (up to a p -adic unit) described in the previous section. In particular, if \mathfrak{p} is the classical height one prime corresponding to f then the image of $L_p(\mathbb{N}, \mathfrak{m}, M)$ in $\mathcal{O}(\mathfrak{p}) \otimes \Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times]$ is $L_{M,\mathfrak{m},f}u$ where u is in $\mathcal{O}(\mathfrak{p})^\times$.

4.3 The one variable p -adic L -function over the Hida family

The first variable of $L_p(\mathbb{N}, \mathfrak{m}, M)$ is our Hida family and the second variable is the cyclotomic variable, which varies over Dirichlet twists and the different critical values $s = 1 \dots k - 1$. In the previous section, we saw that specializing in the first variable recovered the p -adic L -function of [18]. If we specialize in the second variable, we recover a p -adic L -function that interpolates a fixed special value over our Hida family. This L -function won't be meaningful for small classical weights in \mathbb{T} , because the classical cyclotomic p -adic L -function only interpolates s -values less than the weight.

Let χ be a Dirichlet character of level $N = p^r M$ with N prime to p . Then χ gives a homomorphism $\mathbb{Z}/N\mathbb{Z}^\times \rightarrow \mathbb{C}_p$ and a homomorphism $\Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times] \rightarrow \mathbb{C}_p$. The prime $\mathfrak{p}_{\chi,s}$ of $\Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times]$ that is the kernel of this map corresponds to twisting the L -function by χ and evaluating that L -function at $s = 1$.

The image of $L_p(\mathbb{N}, \mathfrak{m}, M)$ along the map $\mathbb{T}_{N,\mathfrak{m}} \otimes \Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times] \rightarrow \mathbb{T}_{N,\mathfrak{m}} \Lambda_K[\mathbb{Z}/Mp\mathbb{Z}^\times] / \mathfrak{p}_{\chi,s}$ gives a one variable p -adic L -function $L_p(\mathbb{N}, \mathfrak{m}, \chi) \in \mathbb{T}_{N,\mathfrak{m}} \otimes \mathcal{O}_K[\chi]$. This L -function interpolates the values of the L -functions of the cusp forms in our Hida family for a fixed twist.

Theorem 4.4. *Let $\mathfrak{p} \in \text{Spec}(\mathbb{T}_{N,\mathfrak{m}})$ be a classical height one prime corresponding to the modular form f . The image of $L_p(\mathbb{N}, \mathfrak{m}, \chi)$ in $\mathbb{T}_{N,\mathfrak{m}}/\mathfrak{p}$ is $L^{alg}(f, \chi, 1)u$, where u is a p -adic unit.*

5 Some Geometric Preliminaries

Let C_1 and C_2 two connected components of $\text{Spec}(\mathbb{T}_{N,\mathfrak{m}})$. It is often the case that structure maps (i.e. the map onto the weight space) π_i from C_i to $\text{Spec}(\Lambda)$ are isomorphisms. This is an ideal situation, as functions on $\text{Spec}(\Lambda_K)$ are easily understood through the Weierstrass preparation theorem. However, there are many examples of components whose structure maps are not isomorphisms (e.g. families of CM forms where the class group of the imaginary quadratic field is divisible by some power of p). In this section we address this phenomena by finding formal models of small affinoid neighborhoods of C_i^{rig} whose coordinate ring is isomorphic to $\mathcal{O}_K\langle T \rangle$. We choose these formal models in a way so that they still carry information about congruences between cusp forms. We address this in the first subsection. In the second subsection we introduce an auxiliary p -adic metric on the \mathcal{O}_K points of a schemes over \mathcal{O}_K . There is nothing particularly novel here, but it will be convenient for later arguments. Finally, we define intersection multiplicities and explain how we can pass from schemes to rigid varieties. There is no extra difficulty for us to work in a general geometric situation and we do so.

Throughout this section we set $\pi : X \rightarrow \text{Spec}(\Lambda_K)$ be a finite morphism. We will assume that X is affine with coordinate ring R . By a component of X we refer to a subscheme $C := \text{Spec}(R/\mathfrak{a})$ of X where \mathfrak{a} is a minimal ideal of X .

5.1 The inverse function theorem for formal models

Let C be a component of X and let $x \in C$ be a \mathcal{O}_K point. We will assume that π is etale at x .

Lemma 5.1. *There is an affinoid neighborhood U of x in C^{rig} and V of $\kappa := \pi(x)$ in Λ^{rig} such that U and V are isomorphic as rigid varieties. In particular, if \mathfrak{p}_x is the maximal ideal defining x , the affinoid U may be taken to be $\{y \in C \mid |f(y)| \leq \epsilon, y \in \mathfrak{p}_x\}$.*

Proof. This is a rigid analytic inverse function theorem. It can be deduced from the results of Chapter III section 9 in [21]. \square

Since C is finite over $\text{Spec}(\Lambda)$, we may write $C = \text{Spec}(A)$ with $A = \mathcal{O}_K[[T_0]][T_1, \dots, T_n]/I$. After a change of variables, we may assume that x corresponds to the point $T = T_i = 0$. By Lemma 5.1 there exists $N > 0$ such that the affinoid variety $X = \text{Sp}((A \otimes K)\langle \frac{1}{p}^N T_i \rangle)$ maps isomorphically onto $\text{Sp}(K\langle \frac{1}{p}^N T_0 \rangle)$. Set $Y_i = \frac{1}{p}^N T_i$ so that $\text{Spec}(A\langle Y_i \rangle)$ is a formal model for X . Then $Y_i = f_i(Y_0)$ for some $f_i \in K\langle Y_0 \rangle$ (these power series will define the inverse map $\text{Sp}(K\langle Y_0 \rangle) \rightarrow X$). The coefficients of f_i tend to zero so we may find k such that $p^k f_i \in \mathcal{O}_K\langle Y_0 \rangle$ for each k . In particular we find $f_i \in \mathcal{O}_K\langle \frac{1}{p}^k Y_0 \rangle$ and we may assume that f_i has all integral coefficients by replacing Y_0 with $\frac{1}{p}^k Y_0$.

We now have an explicit description of X as $\text{Sp}(K\langle Y_i \rangle / (Y_i - f_i(Y_0)))$. The ring $A\langle Y_i \rangle$ injects into $(A \otimes K)\langle Y_i \rangle \cong K\langle Y_i \rangle / (Y_i - f_i(Y_0))$. Since the power series f_i have integral coefficients, we have an isomorphism $A\langle Y_i \rangle \cong \mathcal{O}_K\langle Y_i \rangle / Y_i - f_i(Y_0)$ and the ring $\mathcal{O}_K\langle Y_i \rangle / Y_i - f_i(Y_0)$ is isomorphic to $\mathcal{O}_K\langle Y_0 \rangle$. This is more or less an inverse function theorem for formal models.

Lemma 5.2. *Write $C = \text{Spec}(A)$ with $A = \mathcal{O}_K[[T_0]][T_1, \dots, T_n]/I$ as above. For some $N > 0$ there exists an isomorphism of formal schemes*

$$\text{Spec}(A\langle \frac{1}{p}^N T_i \rangle) \rightarrow \text{Spec}(\mathcal{O}_K\langle \frac{1}{p}^N T_0 \rangle).$$

The isomorphism is given by projecting onto the T_0 coordinate.

The particular situation we are interested in involves two components C_1 and C_2 of $X = \text{Spec}(A)$ corresponding to the minimal primes \mathfrak{a}_1 and \mathfrak{a}_2 . Let x_1 (resp x_2) be a \mathcal{O}_K -point of C_1 (resp C_2)

5.2 p -adic Distances and Congruences

We start by discussing p -adic distances for affine schemes that are quotients of rings of power series. We work in this generality because it adds no extra difficulty. First let's consider an open n -dimensional ball B centered around the origin of radius p^{-1} . The ring of analytic functions converging on this ball is the Tate algebra $\mathcal{O}_K[[p^{-1}T_1, \dots, p^{-1}T_n]]$. If (x_1, \dots, x_n) and (y_1, \dots, y_n) are two \mathcal{O}_K points in B , a natural choice for the distance between them is $\max |x_i - y_i|_p$. It would be great to translate this definition into something more intrinsic. Let A be some reduced quotient of a Tate algebra over \mathcal{O}_K .

Definition 1. *Let R be a ring of (topologically) finite type over \mathcal{O}_K . Let I be an ideal of R . For any prime $\mathfrak{p} \in \text{Spec}(R)$ let $I(\mathfrak{p})$ be the ideal $I + \mathfrak{p} \pmod{\mathfrak{p}}$ in R/\mathfrak{p} . If \mathfrak{p} is a prime that is maximal in $R \otimes \mathbb{Q}_p$, then R/\mathfrak{p} is a discrete valuation ring (it is a finite extension of \mathbb{Z}_p .) Let $|I|_{\mathfrak{p}}$ be the largest absolute value occurring in $I(\mathfrak{p})$, where the absolute value is normalized so $|p|_{\mathfrak{p}} = p^{-1}$.*

Definition 2. *Let \mathfrak{p}_1 and \mathfrak{p}_2 be height one prime ideals with residue characteristic 0. We define $d(\mathfrak{p}_1, \mathfrak{p}_2)$ to be $|I|_{\mathfrak{p}_1} |I|_{\mathfrak{p}_2}$.*

Lemma 5.3. *The following properties of $d(,)$ hold:*

1. $d(\mathfrak{p}_1, \mathfrak{p}_2) = d(\mathfrak{p}_2, \mathfrak{p}_1)$
2. $d(\mathfrak{p}_1, \mathfrak{p}_1) = 0$
3. Let (f_1, \dots, f_r) be any set of generators of \mathfrak{p}_1 . Then $d(\mathfrak{p}_1, \mathfrak{p}_2) = \max |f_i \bmod \mathfrak{p}_2|_p$.
4. Suppose $A = \mathcal{O}_K[[T_1, \dots, T_n]]$. Assume also that \mathfrak{p}_1 corresponds to (x_1, \dots, x_n) and \mathfrak{p}_2 corresponds to (y_1, \dots, y_n) . Then $d(\mathfrak{p}_1, \mathfrak{p}_2) = \max |x_i - y_i|_p$.
5. Suppose we have a closed embedding $f : \text{Spec}(A) \rightarrow \text{Spec}(\mathcal{O}_K[[T_1, \dots, T_n]])$ (so that A is a quotient of a Tate algebra.) Then $d(\mathfrak{p}_1, \mathfrak{p}_2) = d(f(\mathfrak{p}_1), f(\mathfrak{p}_2))$.
6. The non-Archimedean triangle inequality holds. That is $d(\mathfrak{p}_1, \mathfrak{p}_3) \leq \max(d(\mathfrak{p}_1, \mathfrak{p}_2), d(\mathfrak{p}_2, \mathfrak{p}_3))$.

Since $\mathcal{O}_K[[T_1, \dots, T_n]]$ is local with maximal ideal (π_K, T_1, \dots, T_n) , we see that A is also local with a maximal ideal \mathfrak{m} . There is an equivalent definition of distance, which easily connects to congruences of cusp forms.

Lemma 5.4. *Suppose*

$$A/\mathfrak{m} \cong \mathcal{O}_K \cong A/\mathfrak{p}_2.$$

Let r be the largest integer such that $\mathfrak{m}^r + \mathfrak{p}_1 = \mathfrak{m}^r + \mathfrak{p}_2$. Then $p^{-\frac{r}{e}} = d(\mathfrak{p}_1, \mathfrak{p}_2)$, where e is the ramification index of \mathcal{O}_K over \mathbb{Z}_p . In particular, the natural map

$$A \rightarrow A(\mathfrak{p}_1) \cong \mathcal{O}_K \rightarrow \mathcal{O}_K/\pi_K^r$$

is the same as

$$A \rightarrow A(\mathfrak{p}_2) \cong \mathcal{O}_K \rightarrow \mathcal{O}_K/\pi_K^r,$$

and r is the largest integer for which this is true.

Proof. Not very hard. □

Now consider the big Hecke algebra \mathbb{T}_N described in 4.1. Let \mathfrak{m} be a maximal ideal of \mathbb{T}_N . The ring $\mathbb{T}_{N,\mathfrak{m}}$ is local, reduced, finite over Λ and \mathfrak{m} -adically complete. We will assume that the residue fields of $\mathbb{T}_{N,\mathfrak{m}}$ and Λ are the same, which is equivalent to saying $\mathbb{T}_{N,\mathfrak{m}}/\mathfrak{m} \cong \mathcal{O}_K/\pi_K \mathcal{O}$ (if this is not the case we may replace K with a larger extension). Let r_1, \dots, r_n generate $\mathbb{T}_{N,\mathfrak{m}}$ as a Λ algebra. If r_i is a unit, then it is equivalent to an element of $\mathcal{O}_K\langle T_0 \rangle$ modulo \mathfrak{m} , so we may assume that each r_i is in \mathfrak{m} . Thus the r_i are topologically nilpotent and we have a surjection $\mathcal{O}_K[[T_1, \dots, T_n]]$ by sending T_i to r_i . Geometrically, we can embed the local components of our big Hecke algebra into an n -dimensional open unit Ball. Summarizing everything gives:

Proposition 1. *The local big Hecke algebra $\mathbb{T}_{N,\mathfrak{m}}$ embeds into an n -dimensional open unit ball. This mapping is "isometric" with respect to the natural p -adic metric on the p -adic ball and the metric $d(,)$ defined above. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in \text{Spec}(\mathbb{T}_{N,\mathfrak{m}})$ be \mathcal{O}_K -points. If $d(\mathfrak{p}_1, \mathfrak{p}_2) = p^{-\frac{r}{e}}$ then the modular forms (not necessarily of classical weight) corresponding to \mathfrak{p}_i are congruent modulo π_K^r but not π_K^{r+1} .*

5.3 Intersection Multiplicities

In this section we will define the intersection multiplicity of two crossing components of a curve over \mathcal{O}_K . After proving some basic properties we will reduce our problem to talking about $f(C_i)$ as in the previous section. Let X be a scheme over \mathcal{O}_K such that X^{rig} has dimension one. Let C_1 and C_2 be connected components of X and let x be an \mathcal{O}_K point of X whose residue characteristic is 0 (so we may think of x as a point in X^{rig} .) Let $j_i : C_i \rightarrow X$ be the natural inclusion. Let \mathcal{I}_i be the \mathcal{O}_X -sheaf of ideals that define the component C_i .

Definition 3. *The intersection multiplicity $I(X, C_1, C_2, x)$ of C_1 and C_2 at x is the K dimension of*

$$(\mathcal{O}_{C_1}/j_1^*(\mathcal{I}_2))_x = (\mathcal{O}_X/(\mathcal{I}_1 + \mathcal{I}_2))_x = (\mathcal{O}_{C_2}/j_2^*(\mathcal{I}_1))_x.$$

Remark. *The intersection multiplicity is nonzero if and only if both C_1 and C_2 contain x .*

Since this definition is Zariski local (and rigid analytic local, we shall see...) we may take an affine neighborhood $U = \text{Spec}(A)$ of x . Let \mathfrak{a}_1 and \mathfrak{a}_2 be the minimal prime ideals of A defining the components C_1 and C_2 . Let p_x be the prime corresponding to x . Then

$$I(X, C_1, C_2, x) = \dim_{\mathbb{Q}_p}(A/(\mathfrak{a}_1 + \mathfrak{a}_2))_{p_x}.$$

Lemma 5.5. *The following properties of intersection multiplicities are true.*

1. *Let X' be another formal model of X^{rig} . Let C'_1 and C'_2 be connecting components corresponding to C_1^{rig} and C_2^{rig} . These components cross at x' whose image in X^{rig} is x . Then $I(X, C_1, C_2, x) = I(X', C'_1, C'_2, x)$. In other words, the intersection multiplicity only depends on the rigid fiber.*
2. *Recall how stalks are defined for a sheaf on a rigid analytic varieties*

$$\mathcal{F}_{X^{rig}, x} = \varinjlim_{\substack{x \in U \\ U \text{ is affinoid}}} \mathcal{F}(U).$$

Then $I(X, C_1, C_2, x) = \dim_{\mathbb{Q}_p}(\mathcal{O}_{X^{rig}}/(\mathcal{I}_1^{rig} + \mathcal{I}_2^{rig}))$. In other words, we can use the rigid analytic stalks or the Zariski stalks to find intersection numbers.

3. *Let U be an affinoid neighborhood of x . Let \mathfrak{U} be a formal model of U and let \mathfrak{C}_i be the formal models of $U \cap C_i^{rig}$ that are components of \mathfrak{U} . Then $I(X, C_1, C_2, x) = I(\mathfrak{C}_1, \mathfrak{C}_2, x)$.*
4. *Let $i : Y \rightarrow X$ be a closed subscheme such that $i(Y)$ contains the generic points of C_1 and C_2 . Then $I(X, C_1, C_2, x)$ is the same as $I(Y, Y \times_X C_1, Y \times_X C_2, i^{-1}(x))$.*

Proof. The first statement is true because p is invertible in $\mathcal{O}_{X, x}$. The third statement is an immediate consequence of the second statement. To prove the second statement, pick an affine neighborhood of x and use the fact that $\widehat{\mathcal{O}_{X, x}} \cong \widehat{\mathcal{O}_{X^{rig}, x}}$, where \hat{A} denotes the completion of a local ring A along it's maximal ideal (see [3].) The last statement is easily checked by picking an affine neighborhood of x .

□

In the simple situation of two rational curves crossing in $\mathcal{O}_K\langle T_0, T_1 \rangle$, we can come up with a precise formula for the intersection number. Let X be finite over $\text{Spec}(\mathcal{O}_K\langle T_0 \rangle)$ and let $X \rightarrow \mathcal{O}_K\langle T_0, T_1 \rangle$ be a closed embedding. Let C_1 and C_2 be two connected components of X that are isomorphic both $\mathcal{O}_K\langle T_0 \rangle$. Then we have embeddings $C_i \rightarrow \text{Spec}(\mathcal{O}_K\langle T_0, T_1 \rangle)$ that factor through X and we see that $C_i = \text{Spec}(\mathcal{O}_K\langle T_0 \rangle T_1 / (T_1 - f_i(T_0)))$ with $f_i(T_0) \in \mathcal{O}_K\langle T_0 \rangle$. Then we can compute the intersection number:

Lemma 5.6. *The intersection $I(X, C_1, C_2, x)$ is equal to the largest power of T_0 dividing $f_1 - f_2$.*

6 Crossing components in Hida families

We are now ready to prove Theorem 1.2. Recall our definition of $\mathbb{T}_{N,m}$ from Section 4.1 as a local component of the Hecke algebra that acts on the space of all cusp forms of tame level N . There is a map

$$\pi : \text{Spec}(\mathbb{T}_{N,m}) \rightarrow \text{Spec}(\Lambda_K) \cong \text{Spec}(\mathcal{O}_K[[T]]).$$

We will assume that $\text{Spec}(\mathbb{T}_{N,m})$ has at least two components C_1 and C_2 . Let κ be a \mathcal{O}_K -point of $\text{Spec}(\Lambda_K)$ that is the p -adic limit of classical weights and let x_1 (resp x_2) be a \mathcal{O}_K -point of C_1 (resp. C_2) with $\pi(x_1) = \kappa$. After a change of variables we may take κ to be the point $T = 0$. We will assume that the restriction of π to C_1 (resp C_2) is etale at x_1 (resp x_2). If C_1 and C_2 cross above κ then it is possible to choose x_1 and x_2 to be the same point when viewed as points of $\text{Spec}(\mathbb{T}_{N,m})$.

6.1 Reducing to the simplest geometric situation

We may write $C_i = \text{Spec}(A_i)$ where A_i is generated as a Λ_K -algebra by $T_{i,1}, \dots, T_{i,n}$ and we may choose the $T_{i,1}$ so that x_i corresponds to the origin. By applying Lemma 5.2 to C_1 and C_2 simultaneously we know that there exists $N > 0$ such that

$$B_i := A_i \langle \frac{1}{p}^N T_0, \frac{1}{p}^N T_{i,1}, \dots, \frac{1}{p}^N T_{i,n} \rangle \cong \mathcal{O}_K \langle \frac{1}{p}^N T_0 \rangle.$$

Let $Y = \frac{1}{p}^N T_0$. Then $\text{Spec}(B_i)$ is a connected component of $\text{Spec}(\mathbb{T}_{N,m}\langle Y \rangle)$. To see this, consider the commutative diagram

$$\begin{array}{ccc} \text{Spec}(B_i) & \longrightarrow & \text{Spec}(\mathbb{T}_{N,m}\langle Y \rangle) \\ \pi_i^{-1} \uparrow & & \downarrow \\ \text{Spec}(\mathcal{O}_K\langle Y \rangle) & \xlongequal{\quad} & \text{Spec}(\mathcal{O}_K\langle Y \rangle) \end{array}$$

From this diagram we see that the generic point of $\text{Spec}(B_i)$ must be sent to a minimal prime in $\text{Spec}(\mathbb{T}_{N,m}\langle \frac{1}{p}^N Y \rangle)$ and that the map $\mathbb{T}_{N,m}\langle \frac{1}{p}^N Y \rangle \rightarrow B_i$ is surjective. Let Z be the scheme theoretic union of $\text{Spec}(B_i)$ inside of $\text{Spec}(\mathbb{T}_{N,m}\langle \frac{1}{p}^N Y \rangle)$. Then Z comes naturally equip with a map to $\text{Spec}(\mathcal{O}_K\langle \frac{1}{p}^N Y \rangle)$, which we call π by abuse of notation. This map is surjective and finite of degree two by our definition of Z . Thus

$$Z = \text{Spec}(\mathcal{O}_K \langle \frac{1}{p}^N Y \rangle [T] / f(T, Y)),$$

where T is some indeterminate and f is monic of degree two in T . As π admits two sections (one for each $\text{Spec}(B_i)$) the polynomial f factors into linear terms, i.e.

$$f(T, Y) = (T - g_1(Y))(T - g_2(Y)).$$

Here we have $g_i \in \mathcal{O}_K\langle \frac{1}{p}^N Y \rangle$ and g_i corresponds to the closed subscheme $\text{Spec}(B_i)$. The points x_1 and x_2 are the same if and only if g_1 and g_2 have the same constant term.

There is a natural map $s : Z \rightarrow \mathbb{T}_{N, \mathfrak{m}}$. For any two \mathcal{O}_K -points y_1, y_2 in Z we have a relation between the distances of these points in the two spaces:

$$d(y_1, y_2) = p^N d(s(y_1), s(y_2)).$$

These points correspond to cusp forms f_{y_1} and f_{y_2} . We combine Lemma 1 with this relation to get:

Lemma 6.1. *If $d(y_1, y_2) = p^{\frac{-r}{e}-N}$ then the cusp forms f_{y_1} and f_{y_2} are congruent modulo π_K^r but not π_K^{r+1} . Informally, the distances between points in Z tells us exactly how congruent the corresponding cusp forms are.*

6.2 An ideal of differences of L -values

Recall that for χ and $k \in \mathbb{N}$ there is a 1-variable p -adic L -function $L_p(\mathbb{T}_{N, \mathfrak{m}}, \chi) \in \mathbb{T}_{N, \mathfrak{m}}[\chi]$. Let $L_p(B_i, \chi)$ be the restriction of this function to $\text{Spec}(B_i)$. In particular $L_p(B_i, \chi)$ is in B_i . Then for $x \in \text{Spec}(B_i)$ corresponding to a classical modular form f_x we see that $L_p(B_i, \chi)$ evaluated at x is equal to the algebraic part of $L(f_x, \chi, 1)$. The isomorphism $\pi_i : \text{Spec}(B_i[\chi]) \rightarrow \text{Spec}(\mathcal{O}_K[\chi]\langle \frac{1}{p}^N T \rangle)$ allows us to view $L_p(B_i, \chi)$ as a power series in T . In fact, this gives us the Taylor series expansion of $L_p(C_i, \chi)$ expanded around x_i .

Definition 4. *Let I_L be the ideal of $\mathcal{O}_K^{cyc}\langle \frac{1}{p}^N T \rangle$ generated by the elements $L_p(B_1, \chi) - L_p(B_2, \chi)$ for all Dirichlet characters χ .*

Lemma 6.2. *Let $\kappa \in \text{Spec}(\mathbb{Z}_p\langle T \rangle)$ corresponds to a classical weight. Then $v_p(I_L(\kappa))$ is equal to the valuation of $g_1(\kappa) - g_2(\kappa) - N$.*

Proof. Let y_i be the point of $\text{Spec}(B_i)$ lying above κ . By Proposition 3.5 we know that $v_p(I_L(\kappa))$ tells us exactly how congruent the modular forms f_{y_i} associated to these points are. Then by applying Lemma 6.1 we see that $v_p(I_L(\kappa)) = d(y_1, y_2)p^{-N}$. The coordinates of y_i are $(\kappa, g_i(\kappa))$. Then applying parts four and five of Lemma 5.3 we find that $d(y_1, y_2) = |g_1(\kappa) - g_2(\kappa)|$. The result follows. \square

6.3 Proof of Theorem 1.2

We are now ready to prove Theorem 1.2 by comparing $g_1(Y) - g_2(Y)$ with the ideal I_L . The proof involves using Lemma 6.2 to see how congruences behave as we approach the crossing point at classical weights.

Definition 5. *Define B_{p^r} to be $\text{Spec}(\mathbb{Z}_p\langle \frac{1}{p}^r Y \rangle)$, the neighborhood of radius $\frac{1}{p}^r$ around $\kappa = 0$.*

Lemma 6.3. *Let $f(T) \in \mathbb{Z}_p\langle Y \rangle$. Let $x \in \mathbb{Z}_p$ with $v_p(x) < 1$. If $f(x) \neq 0$ (equivalently $T - x$ does not divide $f(T)$), then there exists a ball $B_{p^r}(x)$ centred at x such that $f(T)$ is equal to a power of p times a unit when restricted to $B_{p^r}(x)$. Equivalently, we have $f(T) = p^s u(T)$ with $u(T)$ being in $\mathbb{Z}_p[[p^r(T - x)]]^\times$. Any neighbourhood of x where $f(T)$ has no roots will suffice.*

Proposition 2. *The largest power of Y dividing the ideal I_L is $I(\mathbb{T}_N, C_1, C_2, x_1)$.*

Proof. Let $n = I(\mathbb{T}_N, C_1, C_2, x)$ and let m be the largest power of Y contained in I_L . Let χ be a Dirichlet character and $k \in \mathbb{N}$. By Lemma 6.3 and Lemma 5.6 we may replace $\mathcal{O}_K\langle Y \rangle$ with a smaller ball $B_{p^r}(0)$ where

$$g_1(Y) - g_2(Y) = Y^n u(Y) \pi_K^r \quad (3)$$

$$L_p(B_1, \chi) - L_p(B_2, \chi) = Y^{m_{\chi,s}} v_{\chi,s}(Y) c_{\chi,s}. \quad (4)$$

Here $u(Y), v_{\chi,s}(Y)$ are units and $c_{\chi,s}$ is a constant in $\mathcal{O}_K[\chi]$. Note that $m \min(m_{\chi,s})$. Pick a sequence t_n of points in $B_{p^r}(0)$ that converge to 0 such that each t_n corresponds to a classical weight. We denote by $L_p(B_i, \chi)(t_n)$ the function $L_p(B_i, \chi)$ evaluated at the point t_n . Then by Lemma 6.2 we know that

$$v_p(g_1(t_n) - g_2(t_n)) \leq v_p(L_p(B_1, \chi)(t_n) - L_p(B_2, \chi)(t_n)) \text{ which gives} \quad (5)$$

$$nv_p(t_n) + r \leq mv_p(t_n) + t. \quad (6)$$

As $v_p(t_n) \rightarrow \infty$ as $n \rightarrow \infty$ we see that $n \leq m_{\chi,s}$ and so $n \leq m$. In particular we find that Y^n divides I_L . Conversely, we know

$$mv_p(t_n) \leq m_{\chi,s} v_p(t_n) \leq v_p(L_p(B_1, \chi)(t_n) - L_p(B_2, \chi)(t_n)).$$

Applying Lemma 6.2 again while $n \rightarrow \infty$ shows $m \leq n$. □

The proof of Theorem 1.2 follows as a corollary.

7 Some examples

In this section we explore two situations where crossings may occur. First, we look at two Hida families of different levels with the same residual representation. Under a suitable hypothesis on the levels, we can determine if the two families will cross in a higher level by looking at the L -functions on each family modified by appropriate Euler factors. These modified L -functions were introduced in [7] for trivial tame character using results from [24].

7.1 Components of different level

In this subsection we apply our results on crossing components to the situation described in Section 2.6 of [7]. We will briefly summarize the set up. For details and references see [7]. Let $\bar{\rho}$ be a modular residual Galois representation and let \bar{V} be the \mathbb{F}_q vector space on which $G_{\mathbb{Q}}$ acts. We will assume that $\bar{\rho}$ is odd, irreducible, p -ordinary, and p -distinguished. Fix a p -stabilization of $\bar{\rho}$. We may assume that \mathbb{F}_q is the possible extension of \mathbb{F}_p (i.e. \mathbb{F}_q is generated by the traces of $\bar{\rho}$). Let $N(\bar{\rho})$ be

the conductor of $\bar{\rho}$. For a prime $l \neq p$ let n_l be the dimension of the I_l invariant of \bar{V} . Let Σ be a finite set of primes not containing p . Define

$$N(\Sigma) := N(\bar{\rho}) \prod_{l \in \Sigma} l^{n_l}.$$

For any tame level N we let \mathbb{T}'_N be the Hecke algebra acting on $S(Np^\infty, \mathcal{O}_K)$ generated by U_p and T_l for all $l \nmid Np$ (explicitly, we are just leaving out the Atkin Lehner operators).

We let $\mathbb{T}_N^{\text{new}}$ denote the Hecke algebra generated by T_l for primes $l \nmid Np$ and U_l for $l \mid Np$ acting on the subspace of $S(Np^\infty, \mathcal{O}_K)$ consisting of all newforms. Then we have a natural map of Λ -algebras

$$\mathbb{T}'_{N(\Sigma)} \rightarrow \Pi_{M|N(\Sigma)} \mathbb{T}_M^{\text{new}}.$$

This map becomes an isomorphism after tensoring over Λ with its fraction field \mathcal{L} . As described by Hida [11] there is a Galois representation $\rho'_M : G_{\mathbb{Q}} \rightarrow \mathbb{T}_M^{\text{new}} \otimes \mathcal{L}$ for any M . This gives a Galois representation $\rho' : G_{\mathbb{Q}} \rightarrow \mathbb{T}'_{N(\Sigma)} \otimes \mathcal{L}$. We have the following two theorems

Theorem 7.1. *There exists a unique maximal prime \mathfrak{m} of $\mathbb{T}'_{N(\Sigma)}$ such that the residual representation of the composition*

$$G_{\mathbb{Q}} \rightarrow \mathbb{T}'_{N(\Sigma)} \rightarrow \mathbb{T}_{N(\Sigma), \mathfrak{m}}$$

is $\bar{\rho}$. Furthermore, there is a unique maximal prime \mathfrak{n} of $\mathbb{T}_{N(\Sigma)}$ such that the two local Hecke algebras are isomorphism:

$$\mathbb{T}_{N(\Sigma), \mathfrak{n}} \cong \mathbb{T}'_{N(\Sigma), \mathfrak{m}}.$$

Proof. Cite Wiles and Diamond. □

Theorem 7.2. *Let \mathfrak{a} be a minimal primes ideal of $\mathbb{T}_{N(\Sigma), \mathfrak{n}}$. There exists some $N(\mathfrak{a}) \mid N(\Sigma)$ and a minimal prime ideal \mathfrak{a}' of $\mathbb{T}_{N(\mathfrak{a})}^{\text{new}}$ that makes the following diagram commute.*

$$\begin{array}{ccc} \mathbb{T}_{N(\Sigma), \mathfrak{n}} \cong \mathbb{T}'_{N(\Sigma), \mathfrak{m}} & \longrightarrow & \Pi_{M|N(\Sigma)} \mathbb{T}_M^{\text{new}} \\ \downarrow & & \downarrow \\ \mathbb{T}_{N(\Sigma), \mathfrak{n}} / \mathfrak{a} & \longrightarrow & \mathbb{T}_{N(\mathfrak{a})}^{\text{new}} / \mathfrak{a}' \end{array}$$

Proof. This follows from 7.1 and the isomorphism

$$\mathbb{T}'_{N(\Sigma)} \otimes \mathcal{L} \rightarrow \Pi_{M|N(\Sigma)} \mathbb{T}_M^{\text{new}} \otimes \mathcal{L}.$$

See Proposition 2.5.2 in [7] for more details. □

Remark. *We may think of $\text{Spec}(\mathbb{T}_{N(\Sigma), \mathfrak{n}} / \mathfrak{a})$ as a family of old forms of level $N(\Sigma)$ and $\text{Spec}(\mathbb{T}_{N(\mathfrak{a})}^{\text{new}} / \mathfrak{a}')$ as a family of new forms of level $N(\mathfrak{a})$. If $x \in \text{Spec}(\mathbb{T}_{N(\Sigma), \mathfrak{n}} / \mathfrak{a})$ corresponds to the classical old form f_x , then there is a corresponding $x' \in \text{Spec}(\mathbb{T}_{N(\mathfrak{a})}^{\text{new}} / \mathfrak{a}')$ that is sent to x under the map $\text{Spec}(\mathbb{T}_{N(\mathfrak{a})}^{\text{new}} / \mathfrak{a}') \rightarrow \text{Spec}(\mathbb{T}_{N(\Sigma), \mathfrak{n}} / \mathfrak{a})$. The point x' corresponds to a newform $f_{x'}$ of level $N(\mathfrak{a})$. The Fourier coefficients of f_x and $f_{x'}$ agree away from the primes dividing the level $N(\Sigma)$.*

By Theorem 1.2 we can determine when two components of $\mathbb{T}_{N(\Sigma)}$ by looking at p -adic L -functions on each component. It is then natural to ask if we can determine when a family of newforms of level $M_1|N(\Sigma)$ will cross a family of newforms of level $M_2|N(\Sigma)$ by looking at p -adic L -functions. To employ Theorem 1.2 it is necessary to relate our p -adic L -functions on $\text{Spec}(\mathbb{T}_{N(\Sigma),n}/\mathfrak{a})$ to our p -adic L -functions on $\text{Spec}(\mathbb{T}_{N(\mathfrak{a}),n}/\mathfrak{a}')$. The former interpolates special values of eigenforms for the Hecke algebra $\mathbb{T}_{N(\Sigma)}$ and the later interpolates special values of eigenforms for the Hecke algebra $\mathbb{T}_{N(\mathfrak{a})}$. As these two Hecke algebras only differ at $l|N(\Sigma)$, it is natural to suspect that the two L -functions will be the same after introducing some Euler factors for the primes $l|N(\Sigma)$.

Definition 6. Let $l \neq p$ be a prime and let χ be a Dirichlet character of level Mp^r . Define $E_{N(\mathfrak{a})}(\chi, l) \in \mathbb{T}_{N(\mathfrak{a})}$ as follows:

$$E_{N(\mathfrak{a})}(\chi, l) := \begin{cases} 1 - \chi(l)T_l l^{-1} + \chi(l^2)\langle l \rangle l^{-3} & \text{if } l \nmid N(\mathfrak{a}) \\ 1 - \chi(l)T_l l^{-1} & \text{if } l|N(\mathfrak{a}) \end{cases}$$

We then define

$$E_{\Sigma}(\mathfrak{a}, \chi) := \prod_{l \in \Sigma} E_{N(\mathfrak{a})}(\chi, l).$$

Remark. This definition is similar to Definition 2.7.1 and 3.6.1 in [7]. The Euler of [7] varies over a branch of the Hida family and the cyclotomic variable, while our definition only varies over the branch. If the conductor of χ is a power of p then our Euler factor is equal to a specialization of the Euler factor defined in [7].

Proposition 3. Let χ be a Dirichlet character. There exists a unit $u \in \mathbb{T}_{N(\mathfrak{a})}/\mathfrak{a}'$ independent of χ Dirichlet character such that

$$L_p(\mathbb{T}_{N(\Sigma)}/\mathfrak{a}, \chi) = E_{\Sigma}(\mathfrak{a}, \chi) L_p(\mathbb{T}_{N(\mathfrak{a})}/\mathfrak{a}', \chi).$$

Proof. The proof follows from the computations in beginning of the proof of Theorem 3.6.2 in [7]. The only difference is that we are specializing in the cyclotomic variable and we allow a nontrivial tame conductor. \square

Theorem 7.3. Let M_1 and M_2 be two integers dividing $N(\Sigma)$. Let \mathfrak{a}_i be a minimal prime ideal of $\mathbb{T}_{M_i}^{\text{new}}$. Let C_1 and C_2 be the components of $\mathbb{T}_{N(\Sigma)}$ corresponding to \mathfrak{a}_1 and \mathfrak{a}_2 . The following are equivalent:

- The components C_1 and C_2 cross at a point x . We assume that each component is etale at x over the weight space and the weight κ of x is the p -adic limit of classical weights.
- There exists a point x_i of $\text{Spec}(\mathbb{T}_{M_i}^{\text{new}})$ over κ and a unit u of Λ such that for all Dirichlet characters χ the value of $u L_p(\mathbb{T}_{M_1}^{\text{new}}/\mathfrak{a}_1, \chi) E_{\Sigma}(\mathfrak{a}_1, \chi)$ evaluated at x_1 is the same as $L_p(\mathbb{T}_{M_2}^{\text{new}}/\mathfrak{a}_2, \chi) E_{\Sigma}(\mathfrak{a}_2, \chi)$ evaluated at x_2 .

Proof. This is a consequence of Proposition 3 and Theorem 1.2. \square

8 Ramification over the weight space

In this section we describe how p -adic L -functions behave when a Hida family is ramified over the weight space. Recall that Λ_K is the ring of power series over \mathcal{O}_K . Let C be a component of $\text{Spec}(\mathbb{T}_N)$. Then C is affine and we let A be the coordinate ring. Informally, the main result of this section says that C has ramified points over Λ_K if and only if there exists an L -function $L_p(C, \chi)$ that acquires singularities after being hit with the differential operator $\frac{d}{dT}$. Here $L_p(C, \chi)$ refers to the L -function defined in Section 4.3 restricted to the component C . The singularities will be at ramified points.

Theorem 8.1. *A regular \mathcal{O}_K -point $x \in C$ is ramified over $\pi(x) \in \text{Spec}(\Lambda_K)$ if and only if there exists a Dirichlet twist such that $\frac{d}{dT}L(C, \chi)$ has a pole at x , where T is a parameter of the weight space. The ramification index of x over $\pi(x)$ is equal to one more than the largest order pole occurring.*

Proof. First let's assume that π is étale at x . Informally, this means that a small neighborhood of x looks just like part of $\text{Spec}(\Lambda_K)$, so taking the derivative with respect to T should not introduce any poles. More precisely, let \widehat{A}_x be the completion along the maximal ideal of the stalk of \mathcal{O}_C at x . Define $\widehat{\Lambda_{K, \pi(x)}}$ to be the completion along the maximal ideal of the stalk of \mathcal{O}_{Λ_K} at $\pi(x)$. The natural map from $\widehat{\Lambda_{K, \pi(x)}} \rightarrow \widehat{A}_x$ is an étale morphism of complete local rings with isomorphic residue fields. This means the two rings are isomorphic. This isomorphism commutes with the differential operator $\frac{d}{dT}$. In particular there is a map $A \rightarrow \widehat{\Lambda_{K, \pi(x)}}$ such that commutes with $\frac{d}{dT}$ and the maximal ideal of $\widehat{\Lambda_{K, \pi(x)}}$ pulls back to x . It is then clear that for any $f \in A$ the function $\frac{d}{dT}f$ does not have a pole at x .

The converse is more difficult. We begin by making some geometric simplifications similar to those in Section 6.1. Assume that π is ramified at x . After a change of coordinates we may assume that $\pi(x) = 0$. The ring A_x is a discrete valuation ring since x is a regular point of codimension one. Let X be a uniformizing element of A_x . We may assume that X is in A by clearing any denominators. Since X is topologically nilpotent in A we have a map

$$g : C = \text{Spec}(A) \rightarrow \text{Spec}(\mathcal{O}_K[[Y]])$$

induced by the ring map sending Y to X . This map is étale at x so we may apply Lemma 5.2. In particular, let Y_1, \dots, Y_r generate A as a $\mathcal{O}_K[[Y]]$ -algebra. There exists N large enough so that

$$g : \text{Spec}(A\langle \frac{1}{p}^N Y, \frac{1}{p}^N Y_i \rangle) \rightarrow \text{Spec}(\mathcal{O}_K\langle \frac{1}{p}^N Y \rangle)$$

is an isomorphism. Setting $T' = \frac{1}{p}^N T$ and $Y' = \frac{1}{p}^N Y$, we have $T' = f(Y')$ where $f(Y') \in \mathcal{O}_K\langle Y' \rangle$. By increasing N , we may guarantee that the only zero of $f(Y')$ is at $Y' = 0$. Thus $f(Y') = \pi_k^t u(Y') Y'^e$ where $u(Y')$ is a unit. We may write

$$A\langle \frac{1}{p}^N Y, \frac{1}{p}^N Y_i \rangle \cong \mathcal{O}_K\langle T', Y' \rangle / (\pi_k^t u(Y') Y'^e - T').$$

The ramification of x over $\pi(x)$ is seen to be e . We also remark distances of points relate to higher congruences of the corresponding cusp forms. If x_1 and x_2 are two \mathcal{O}_K -points of $A\langle \frac{1}{p}^N Y, \frac{1}{p}^N Y_i \rangle$

corresponding to classical cusp forms f_{x_1} and f_{x_2} that are congruent modulo π_K^r then $d(x_1, x_2) = p^{-\frac{r}{e(K|\mathbb{Q}_p)} - N}$. This is more or less the same as Lemma 6.1.

For a Dirichlet character χ we let $L_p(\chi)$ be the restriction of $L_p(C, \chi)$ to $A\langle Y', T' \rangle[\chi]$. Then we may think of $L_p(\chi)$ as an element of $\mathcal{O}_K\langle Y' \rangle[\chi]$ written as

$$\Sigma c_{i,\chi} Y'^i.$$

Let α be an \mathcal{O}_K -point of our weight space $\text{Spec}(\mathcal{O}_K\langle T' \rangle)$ corresponding to a classical weight. By abuse of notation we will also think of α as an element of \mathcal{O}_K for the parameter T' . Let a_1 and a_2 be distinct $\overline{\mathcal{O}_K}$ -points in $\pi^{-1}(\alpha) \subset A\langle Y', T' \rangle \cong \mathcal{O}_K\langle Y' \rangle$. By abuse of notation we will also think of a_i as an element of $\overline{\mathcal{O}_K}$ for the parameter Y' . Both a_1 and a_2 are roots of $\alpha - \pi_K^t u(Y') Y'^e$ and $d(a_1, a_2) = |a_1 - a_2|_p$. Relating $d(a_1, a_2)$ to congruences and then applying Theorem 3.5 gives

$$\log_p d(a_1, a_2) = v_p(a_1 - a_2) \tag{7}$$

$$= \min_{\chi} (v_p(L_p(\chi)(a_1) - L_p(\chi)(a_2)) + N) \tag{8}$$

$$= \min_{\chi} (-N + v_p(\Sigma c_{i,\chi} (a_1^i - a_2^i))). \tag{9}$$

If we take α to have large enough valuation then both of the a_i 's have valuation larger than $N + 1$ (look at the Newton polygon of $\alpha - \pi_K^t u(Y') Y'^e$). This means $v_p(a_1^i - a_2^i) > v_p(a_1 - a_2) + N + 1$ whenever $i > 1$.

$$v_p(a_1 - a_2) = \min_{\chi} (-N + v_p(\Sigma c_{i,\chi} (a_1^i - a_2^i))) \tag{10}$$

$$= \min_{\chi} (-N + v_p(c_{1,\chi} (a_1 - a_2))) \tag{11}$$

$$= \min_{\chi} (-N + v_p(c_{1,\chi}) + v_p(a_1 - a_2)). \tag{12}$$

Thus there exists a Dirichlet character χ_0 such that $c_{1,\chi_0} \neq 0$. Differentiating the equation

$$T - \pi_K^r u(Y) Y^e = 0$$

with respect to T yields

$$\frac{dY}{dT} = \frac{1}{\pi_K^r Y^{e-1}} \frac{1}{Y u'(Y) + e u(Y)}.$$

This shows that $\frac{dY}{dT}$ has a pole at $Y = 0$ of order $e - 1$. Since $L_p(\chi)$ has a nonzero linear term we find that $\frac{d}{dT} L_p(\chi)$ has a pole of order $e - 1$. □

For the previous result, we choose a parameter for the weight space. The result holds true for any parameter and it would be nice to have a statement that makes no reference to any choice of parameter. This can be achieved using the Gauss-Manin connection, which can be defined without choosing a basis. For an overview of the Gauss-Manin connection see [15] or [16]. More precisely, consider the relative 0-th de Rham cohomology group $H_{dR}^0(C/\text{Spec}(\Lambda_K))$ (see for example [10]). We may identify $H_{dR}^0(C/\text{Spec}(\Lambda_K))$ with $\pi_*(\mathcal{O}_C)$ (here \mathcal{O}_C just denotes the structure sheaf of C). Let U be an open subscheme of C such that $\pi|_U$ is etale. Then following [15] there is a Gauss-Manin connection

$$\nabla : H_{dR}^0(C/\text{Spec}(\Lambda_K))|_U \rightarrow H_{dR}^0(C/\text{Spec}(\Lambda_K)) \otimes \Omega_{\text{Spec}(\Lambda_K)}|_U.$$

If $f \in \Gamma(\mathcal{O}_C, U)$ and T_0 is any parameter of the weight space then $\nabla(f) = \frac{d}{dT_0} f dT_0$. The map ∇ makes sense on all of $\text{Spec}(\Lambda_K)$ when we allow poles in the image.

Corollary 8.2. *Let κ be a \mathcal{O}_K point of Λ_K . The map π is etale at the points above κ if and only if for all Dirichlet characters χ we have $\nabla(L_p(C, \chi)) \in \Gamma(\pi_* \mathcal{O}_C \otimes \Omega_{\text{Spec}(\Lambda_K)}, V)$ where V is some Zariski open containing κ .*

Proof. The "if" direction follows from the existence of the Gauss-Manin connection for smooth maps. For the "only if" let x be a point in $\pi^{-1}(\kappa)$. Let χ be a Dirichlet character. By our hypothesis $\nabla(L_p(C, \chi)) \in (\pi_* \mathcal{O}_C \otimes \Omega_{\text{Spec}(\Lambda_K)})_\kappa$. Note that

$$(\pi_* \mathcal{O}_C \otimes \Omega_{\text{Spec}(\Lambda_K)})_\kappa \cong A_\kappa \otimes_{\Lambda_K, \kappa} \Omega_{\text{Spec}(\Lambda_K), \kappa}.$$

Choose a parameter T_0 of the weight space and let D be the map from $\Omega_{\text{Spec}(\Lambda_K)} \rightarrow \mathcal{O}_{\text{Spec}(\Lambda_K)}$ that sends dT_0 to 1. Then $D \circ \nabla$ is the map $A_\kappa \rightarrow A_\kappa$ given by differentiation with respect to T_0 . There is a natural map $l : A_\kappa \rightarrow A_{(x)}$, the localization of A at x . Since $\nabla(L_p(C, \chi))$ is contained in $(\pi_* \mathcal{O}_C \otimes \Omega_{\text{Spec}(\Lambda_K)})_\kappa$ we see that $l \circ D \circ \nabla(L_p(C, \chi))$ is contained in $A_{(x)}$. This means that $\frac{d}{dT_0} L_p(C, \chi)$ does not have a pole at x . The corollary then follows from Theorem 8.1 □

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